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Liouville Lost, Liouville Regained: Central Charge in a Dynamical Background

S. Carlip*
Department of Physics
University of California
Davis, CA 95616
USA

Abstract

Several recent approaches to black hole entropy obtain the density of states from the central charge of a Liouville theory. If Liouville theory is coupled to a dynamical spacetime background, however, the classical central charge vanishes. I show that the central charge can be restored by introducing appropriate constraints, which may be interpreted as fall-off conditions at a boundary such as a black hole horizon.

email: carlip@dirac.ucdavis.edu

In an effort to understand the microscopic degrees of freedom responsible for black hole entropy, a number of researchers have recently begun to look at Liouville theories, either at spatial infinity [1, 2, 3, 4, 5, 6, 7] or in a neighborhood of the horizon [8]. Such theories naturally arise at the asymptotic boundary of the (2+1)-dimensional BTZ black hole [1], and are relevant to many higher-dimensional black holes whose near-horizon behavior resembles that of the BTZ black hole [9]; they may also be obtained near the horizon of an arbitrary black hole by dimensionally reducing to the r–t plane [8]. Since Liouville theory is a conformal field theory, its density of states can be inferred from its central charge by means of the Cardy formula [10]. There is some debate as to whether the Liouville states represent the genuine gravitational degrees of freedom or merely give an effective "thermodynamic" description [7, 11, 12, 13], but in either case, the result offers a potential explanation for the universality of the Bekenstein-Hawking entropy: the density of states may be determined by conformal symmetry, independent of the details of quantum gravity [14].

The computation of black hole entropy from Liouville theory depends sensitively on the central charge. In particular, it is the appearance of a *classical* central charge—that is, a central term that is already present in the Poisson brackets—that leads to an order $1/\hbar$ contribution to the entropy. Unfortunately, as I shall demonstrate below, when Liouville theory is coupled to a dynamical two-dimensional metric, the classical central charge vanishes. This problem is evaded in some approaches to Liouville theory at spatial infinity, in which the boundary conditions freeze the metric, but it is present in other treatments of spatial infinity [3, 4, 5, 6] and in the near-horizon derivation [8].

Of course, one can recover the central charge, and thus the Bekenstein-Hawking entropy, by freezing the dynamics of the metric. But this seems too strong a condition. The main goal of this paper is to demonstrate that it is sufficient to impose asymptotic fall-off conditions on the metric, either at infinity or at the horizon. To show this, I will introduce a new method for treating such fall-off conditions, via second class constraints and Dirac brackets, which I hope may be more generally useful.

1. Liouville central charge in a dynamic background

We begin with a generalized Liouville action,

$$I_L[\varphi, g] = \frac{1}{4\pi}\int d^2x\sqrt{-g}\left\{\frac{1}{2}g^{ab}\partial_a\varphi\partial_b\varphi + \frac{1}{\gamma}\varphi R + V[\varphi]\right\}. \tag{1.1}$$

For standard Liouville theory, the potential is

$$V[\varphi] = \frac{\mu}{2\gamma^2}e^{\gamma\varphi}, \tag{1.2}$$

but more general forms are possible [15]. After the usual ADM decomposition of the metric,

$$ds^2 = N^2dt^2 - \sigma^2\left(dx + \beta dt\right)^2, \tag{1.3}$$

a tedious but straightforward computation brings the action to the canonical form

$$I_L = \int dt\int dx\ \left(\pi_\sigma\dot\sigma + \pi_\varphi\dot\varphi - N\mathcal{H} - \beta\mathcal{P}\right), \tag{1.4}$$

with canonical momenta

$$
\begin{aligned}
\pi_\sigma &= \frac{1}{2\pi\gamma}\frac{1}{N}\left(\dot\varphi - \beta\varphi'\right), \\
\pi_\varphi &= \frac{1}{4\pi}\frac{\sigma}{N}\left(\dot\varphi - \beta\varphi'\right) + \frac{1}{2\pi\gamma}\frac{1}{N}\left(\dot\sigma - (\beta\sigma)'\right)
\end{aligned}
\tag{1.5}
$$

and Hamiltonian and momentum constraints

$$
\begin{aligned}
\mathcal{H} &= 2\pi\gamma\pi_\varphi\pi_\sigma - \frac{\pi}{2}\gamma^2\sigma\pi_\sigma^2 - \frac{1}{2\pi\gamma}\left(\frac{\varphi'}{\sigma}\right)' + \frac{1}{8\pi}\frac{\varphi'^2}{\sigma} - \frac{1}{4\pi}\sigma V[\varphi], \\
\mathcal{P} &= \varphi'\pi_\varphi - \sigma\pi_\sigma'.
\end{aligned}
\tag{1.6}
$$

(Primes denote derivatives with respect to x, dots derivatives with respect to t.)

The symmetries of the canonical theory are the "surface deformations," generated by

$$
H[\hat\xi] = \int dx\,\hat\xi\mathcal{H}, \quad P[\hat\eta] = \int dx\,\hat\eta\mathcal{P}. \tag{1.7}
$$

These are closely related to diffeomorphisms: given a vector field (ξ^t, ξ^x), the transformation

$$
\delta F = \left\{H[N\xi^t] + P[\xi^x + \beta\xi^t], F\right\} \tag{1.8}
$$

of any function on phase space is equal on shell to the Lie derivative $\mathcal{L}_\xi F$. The surface deformation group splits into two chiral pieces, generated by

$$
H^\pm[\hat\xi] = \frac{1}{2}\left\{H[\sigma\hat\xi] \pm P[\hat\xi]\right\}. \tag{1.9}
$$

Another long but straightforward computation then shows that

$$
\begin{aligned}
\left\{H^\pm[\hat\xi], H^\pm[\hat\eta]\right\} &= \pm H^\pm[(\hat\xi\hat\eta' - \hat\eta\hat\xi')] \\
\left\{H^+[\hat\xi], H^-[\hat\eta]\right\} &= 0.
\end{aligned}
\tag{1.10}
$$

As expected, we have obtained two chiral copies of the Virasoro algebra. But perhaps unexpectedly, instead of the Liouville central charge $c = 12/\gamma^2$, there is no central term in the algebra. Quantization may introduce a central charge, of course, but as noted in the introduction, the quantum central charge has the wrong dependence on $\hbar$ to be useful to explain black hole entropy.

One way to understand what has happened is to write the dynamical metric g_{ab} as

$$
g_{ab} = e^{\gamma\chi}\bar g_{ab}, \tag{1.11}
$$

where $\bar g_{ab}$ is a constant curvature metric, determined by a finite number of moduli. Then using the standard Weyl transformation properties of the scalar curvature,

$$
R[g] = e^{-\gamma\chi}\left(R[\bar g] - \gamma\bar\Delta\chi\right), \tag{1.12}
$$

we find that

$$I_L[\varphi, g] = I_L[\varphi + \chi, \bar{g}] - I_L[\chi, \bar{g}]. \tag{1.13}$$

The dynamical metric thus induces a Liouville action with the opposite sign, and the opposite central charge, of the original action, leading to a cancellation of the central term. This is again a known result, although now "read backwards." It has long been understood that any conformal theory of "matter" coupled to a Liouville theory of "gravity" has vanishing central charge [16]; here, our "matter" is itself a Liouville field, while gravity was treated canonically, but essentially the same arguments apply.

2. Freezing the metric with Dirac constraints

It is clear from eqn. (1.13) that we should be able to recover the standard central charge by freezing the metric g_{ab} (and thus the field χ). This result can, of course, be obtained directly from the action (1.1) by fixing the metric [17]. But for later purposes, it is useful to instead fix the metric by means of constraints in the Poisson algebra.

As our first constraint, we fix the spatial metric σ:

$$C_1(x) = \sigma(x) - \sigma_0(x) = 0 \tag{2.1}$$

where σ_0 is a fixed spatial metric, that is, a (not necessarily constant) metric that has vanishing Poisson brackets with all phase space variables. As our second constraint, we demand that σ_0 be time-independent; from eqn. (1.5), this means that

$$C_2(x) = \pi_\varphi(x) - \frac{\gamma}{2}\sigma\pi_\sigma(x) = 0. \tag{2.2}$$

Note that C_1 and C_2 do not commute. Indeed, from the canonical Poisson brackets,

$$\{C_1(x), C_2(y)\} = -\frac{\gamma}{2}\sigma\delta(x-y). \tag{2.3}$$

In Dirac's terminology, C_1 and C_2 are second class constraints. Such constraints modify the Poisson brackets [18, 19]: one must construct a new bracket $\{\ ,\ \}^*$ to preserve the constraints. Specifically, let $K_{ij}(x,y)$ be the kernel

$$\int dy \sum_j K_{ij}(x,y)\{C_j(y), C_k(z)\} = \delta_{ik}\delta(x-z), \tag{2.4}$$

where $\{\ ,\}$ is the ordinary Poisson bracket. The Dirac bracket is then

$$\{A(x), B(y)\}^* = \{A(x), B(y)\} - \int dz_1 \int dz_2 \sum_{i,j} \{A(x), C_i(z_1)\} K_{ij}(z_1, z_2)\{C_j(z_2), B(y)\}, \tag{2.5}$$

and $\{A(x), C_i(y)\}^*$ is identically zero.

For our Liouville theory, it follows from (2.3) that

$$K_{12}(z_1, z_2) = \frac{2}{\gamma}\sigma^{-1}\delta(z_1 - z_2), \tag{2.6}$$

and it is easy to verify that

$$\begin{aligned}\left\{H^{\pm}[\hat{\xi}], C_1\right\} &= \mp\frac{1}{2}\,(\sigma\xi)' \\ \left\{H^{\pm}[\hat{\xi}], C_2\right\} &= -\frac{1}{4\pi\gamma}\left(\xi' + \frac{\sigma'}{\sigma}\xi\right)' - \frac{1}{8\pi}\xi\sigma^2\left(\frac{dV}{d\varphi} - \gamma V\right).\end{aligned} \tag{2.7}$$

If $V[\varphi]$ is the Liouville potential (1.2), the last term in the bracket of $H^{\pm}$ with C_2 vanishes. The Dirac brackets of the original constraints then become

$$\begin{aligned}\left\{H^{\pm}[\hat{\xi}], H^{\pm}[\hat{\eta}]\right\}^* &\approx \pm H^{\pm}[(\hat{\xi}\hat{\eta}' - \hat{\eta}\hat{\xi}')] \\ &\qquad \pm\frac{1}{4\pi\gamma^2}\int dz\left[\left(\xi' + \frac{\sigma'}{\sigma}\xi\right)\left(\eta' + \frac{\sigma'}{\sigma}\eta\right)' - \left(\eta' + \frac{\sigma'}{\sigma}\eta\right)\left(\xi' + \frac{\sigma'}{\sigma}\xi\right)'\right] \\ \left\{H^{+}[\hat{\xi}], H^{-}[\hat{\eta}]\right\}^* &\approx 0,\end{aligned} \tag{2.8}$$

where, following Dirac, "weak equality" ($\approx$) means "equality up to terms proportional to the constraints." If we now shift $H^{\pm}[\hat{\xi}]$ by appropriate functions of σ,

$$\tilde{H}^{\pm}[\hat{\xi}] = H^{\pm}[\hat{\xi}] + \frac{1}{4\pi\gamma^2}\int dz\xi\left[\left(\frac{\sigma'}{\sigma}\right)^2 - 2\left(\frac{\sigma'}{\sigma}\right)'\right] \tag{2.9}$$

(note that $\{H^{\pm}[\xi], \sigma\}^* \approx 0$), we find

$$\begin{aligned}\left\{\tilde{H}^{\pm}[\hat{\xi}], \tilde{H}^{\pm}[\hat{\eta}]\right\}^* &\approx \pm\tilde{H}^{\pm}[(\hat{\xi}\hat{\eta}' - \hat{\eta}\hat{\xi}')] \pm \frac{1}{4\pi\gamma^2}\int dz\,[\xi'\eta'' - \eta'\xi''] \\ \left\{\tilde{H}^{+}[\hat{\xi}], \tilde{H}^{-}[\hat{\eta}]\right\}^* &\approx 0,\end{aligned} \tag{2.10}$$

describing two commuting Virasoro algebras with classical central charges

$$c^{\pm} = \frac{12}{\gamma^2}, \tag{2.11}$$

thus recovering the standard Liouville result [15].

3. Central charge from near-boundary conditions

While the method used in the preceding section is new, the conclusion, of course, is not. But the method gains power when one realizes that the constraints (2.1)–(2.2) need not be applied globally: it is enough to impose them as fall-off conditions near a boundary.

To see this, it is useful to reinterpret the Dirac brackets (2.5) in a manner suggested by Bergmann and Komar [20]. Consider a system with second class constraints $\{C_i\}$, and let A be a function on phase space that is a candidate for a physical observable. The constraints vanish on the space of physically admissible fields, so if A has nonzero Poisson brackets with any of the C_i, it cannot really be an observable. But we can always shift A

by a term proportional to the constraints without altering the physics. In particular, if we set

$$A^* = A + \sum_i \lambda_i C_i \quad \text{with } \lambda_i = -\sum_j \{A, C_j\} K_{ji} \tag{3.1}$$

with K_{ij} defined as in eqn. (2.4), it is evident that $\{A^*, C_k\} \approx 0$. If we now consider two functions A and B, it is easy to show that the Dirac bracket of A and B is simply the ordinary Poisson bracket of A^* and B^*:

$$\{A, B\}^* = \{A^*, B^*\}. \tag{3.2}$$

Now let us return to our Liouville theory, and suppose that we have a boundary at $x = x_0$. Let us try to impose the constraints (2.1)–(2.2) only in a small neighborhood $[x_0, x_0 + \epsilon]$ of the boundary. The standard Dirac procedure now fails, since $\{C_1, C_2\}$ is no longer invertible, but the Bergmann-Komar variation extends trivially: we need merely add a suitable multiple of the constraints in the region $[x_0, x_0 + \epsilon]$. The Poisson brackets (2.10) then become

$$\begin{aligned} &\left\{\tilde{H}^\pm[\hat{\xi}], \tilde{H}^\pm[\hat{\eta}]\right\}^* \approx \pm\tilde{H}^\pm[(\hat{\xi}\hat{\eta}' - \hat{\eta}\hat{\xi}')] \pm \frac{1}{4\pi\gamma^2}\int_{x_0}^{x_0+\epsilon} dz\, [\xi'\eta'' - \eta'\xi''] \\ &\left\{\tilde{H}^+[\hat{\xi}], \tilde{H}^-[\hat{\eta}]\right\}^* \approx 0, \end{aligned} \tag{3.3}$$

where the only change is that the region of integration in the central term is now restricted to a neighborhood of the boundary. The brackets (3.3) are still those of a pair of commuting Virasoro algebras with central charge (2.11), now defined on a strip. In particular, the central charge is independent of the distance ϵ: the constraints need be imposed only in an arbitrarily small neighborhood of the boundary.

It would be worthwhile to develop a slightly different approach, in which the constraints (2.1)–(2.2) are imposed as fall-off conditions near a boundary: that is, we require $C_1 \sim 0$ and $C_2 \sim 0$ near $x = x_0$, where "~ 0" means "approaches zero sufficiently rapidly." I will not work out details here, but if we choose a coordinate x that is independent of phase space (and thus has vanishing Poisson brackets with all fields), it is clear that the Bergmann-Komar approach can be adapted to conditions $C_i = O((x - x_0)^n)$ to determine an asymptotic algebra of constraints.

4. Future steps: black holes and dilaton gravity

We have now seen that even in a dynamical spacetime background, the central charge of a Liouville theory can be recovered through the imposition of suitable boundary conditions. This means that Solodukhin's "near-horizon conformal field" approach to back hole entropy [8] is consistent after all.

The results above also suggest a new approach to determining L_0, the value of the Virasoro zero mode needed to apply Cardy's formula to the counting of states. A starting point is eqn. (2.9), which tells us that when the near-horizon metric is nonconstant, the

Virasoro generators must be shifted. For conformal coordinates near a black hole horizon,

$$\frac{\sigma'}{\sigma} = \kappa = \frac{2\pi}{\beta}, \tag{4.1}$$

where κ is the surface gravity and β is the inverse Hawking temperature. If we look at diffeomorphisms with period L,

$$\xi_n = \frac{L}{2\pi} e^{2\pi i n x/L} \tag{4.2}$$

(the normalization is fixed by the condition that $\{\xi_m, \xi_n\} = \xi_{m+n}$), eqn. (2.9) yields

$$L_0 = \frac{1}{2\gamma^2}\left(\frac{L}{\beta}\right)^2, \tag{4.3}$$

and thus by Cardy's formula,

$$S = 2\pi\sqrt{\frac{cL_0}{6}} = \frac{2\pi}{\gamma^2}\frac{L}{\beta}. \tag{4.4}$$

Now, in Solodukhin's analysis of dimensionally reduced spherically symmetric gravity in d dimensions* [8],

$$\frac{1}{\gamma^2} = q^2\left(\frac{d-3}{d-2}\right)\frac{A_{\text{hor}}}{8G}, \tag{4.5}$$

where q is an undetermined parameter coming from a field redefinition. This gives the right general form for the Bekenstein-Hawking entropy. If we further demand that

$$c = \frac{3A_{\text{hor}}}{2\pi G}, \tag{4.6}$$

as determined by the boundary analysis of Ref. [21], then (4.4) yields an entropy

$$S = \frac{A_{\text{hor}}}{4G}\frac{L}{\beta}, \tag{4.7}$$

giving the correct Bekenstein-Hawking entropy for the natural choice of periodicity $L = \beta$.

This is clearly not a complete explanation of black hole entropy, since it requires a condition (4.6) that has not been obtained within the Liouville formalism, but it might be more than a mere coincidence. To investigate this issue further, it may be useful to look at a similar analysis of dilaton gravity with more explicitly geometric boundary conditions at the horizon. Work on this problem is in progress.

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*Note that Solodukhin's normalization of the Liouville action differs from the one used here.

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